Practical measurements of unloaded Q_0 for a gap distance of 0.42 cm were carried out using the impedance method,[5] and are shown in Table I together with the corresponding calculated values.

TABLE I
COMPARISON OF CALCULATED AND MEASURED VALUES OF Q_0

Measured frequency in megacycles per second	Tuning position in centimeters	Measured Q_0	Corresponding theoretical Q_0
1500	0.85	1125	4700
1600	1.55	1280	4950
1800	1.8	1720	5100
1870	2.50	900	4400
1690	3.49	575	2100

As can be expected, the measured values of the unloaded Q_0 are about 25 per cent of the theoretical values. This difference is a usual one in case of Q_0 measurements and is due to losses which are not accountable by the simple theory. Losses at the shorting end, imperfect spring contacts, unpolished surfaces, dirt, scratches, capacitance losses at the gap, no losses etc., account for this difference.

V. Conclusion

The first-order theory of back-to-back tuning has been derived and shown to agree reasonably well with the experimental results. Both theory and experiment show that the resonator tuning characteristic obtainable by this method is linear for all practical purposes over a large frequency range. By appropriate design, various frequency ranges and tuning slopes can be obtained. The theoretically expected values of unloaded Q_0 are in the vicinity of 4500, while in actual practice values around 1200 are obtained in the useful range of linear tuning.

The method is, in general, applicable to any form of resonant system where quarter-wave line sections can be employed. For instance, for frequencies in the VHF range, the method can be applied with Lecher lines; in the UHF range, coaxial-type lines are more convenient. The presence of a capacitive gap makes the system especially adaptable where interaction with an electron beam is intended. Due to the simple construction, convenient size, nonharmonic modes, and good mode separation, this method should hold promise of valuable application over a wide frequency spectrum.

VI. Acknowledgment

The authors gratefully acknowledge the valuable help of Dr. H. D. Doolittle of Machlett Laboratories, Inc., Springdale, Conn., for initiating the idea and giving a start to the investigation; and also the assistance of B. Rama Krishna Rao and Ravindra Babu (both of this Institute) in making the many measurements and plots.

Equivalent Circuits for Small Symmetrical Longitudinal Apertures and Obstacles*

ARTHUR A. OLINER†

Summary—**Formulas based on small aperture and small obstacle theory are presented for the determination of equivalent circuits for symmetrical longitudinal apertures and obstacles. These formulas are then applied to several examples of practical interest, including aperture discontinuities in trough waveguide and an obstacle array of interest to anisotropic radomes.**

I. Introduction

THE evaluation of equivalent circuits for waveguide discontinuities often involves the solution of a boundary value problem of considerable complexity. For the class of so-called "small" apertures and obstacles, however, this evaluation becomes particularly simple when the problem is properly phrased. A small aperture or obstacle is one which is located far from the guide walls and whose dimensions are small compared to a wavelength. Under these conditions, the distortion of the field lines in the vicinity of such a small aperture or obstacle, due to some specified excitation, is essentially independent of the cross-sectional shape of the containing waveguide, and depends only on the nature of the excitation and the physical shape of the aperture or obstacle. The quantity which characterizes the aperture or obstacle and which is a function only of its physical geometry and the type of incident excitation is the *polarizability*; since the aperture or obstacle is small compared to wavelength, the polarizability may be determined under static conditions. The function of small aperture or obstacle theory is then to relate the polarizability to the location of the aperture or obstacle within the containing waveguide and to the appropriate

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† Microwave Research Institute, Polytechnic Institute of Brooklyn, Brooklyn, N. Y.

incident mode in a fashion such that the equivalent circuit parameters may be readily evaluated. The value of this method lies not only in the simple phrasing of the problem that is permitted, but also in the fact that the resulting solutions are useful considerably beyond the strict limitations indicated in the preceding.

Small aperture theory was first formulated by Bethe,[1,2] and was soon afterwards rephrased by Marcuvitz[3] in a more compact form particularly suitable for equivalent circuit evaluations. Transverse apertures (*i.e.*, apertures located in a metal plate which is coincident with a waveguide cross-section plane) have been treated by many workers, both with regard to the power coupled through the hole and in terms of its shunt equivalent circuit. It has also been long recognized that the shunt equivalent circuit for a small transverse obstacle may be obtained by Babinet equivalence considerations from the results for a corresponding small transverse aperture.

For *longitudinal* apertures (located in a top or side wall of a waveguide and employed to couple power between two neighboring waveguides) the picture is not as complete. Bethe[2] presents formulas for the power coupled between two adjacent waveguides, and these formulas have been used, for example, in directional coupler designs. Expressions for the scattering matrix elements in multimode waveguide have also been derived[4,5] via simple extensions of Bethe's work. Marcuvitz[6] presents equivalents circuits which were obtained by small aperture methods for several examples of waveguides coupled by longitudinal apertures. Despite the existence of these solutions, general small aperture formulas for the equivalent circuit parameters do not appear anywhere and are not available.

In Section II-B of this paper, general expressions are presented for the equivalent circuit parameters of a certain class of small longitudinal apertures, these expressions being deduced from a knowledge of the scattering matrix elements. The longitudinal apertures considered here are restricted to symmetrical apertures coupling identical waveguides; this restricted class applies to a wide number of cases of practical interest, as is indicated later.

In contrast to the case of longitudinal apertures, small obstacle theory applicable to *longitudinal obstacles* has not been exploited heretofore. Formulas are presented in Section II-C of this paper which permit the ready evaluation of the equivalent circuits of small longitudinal obstacles which are symmetrical.[7] Since it is felt that the derivations of these expressions and those for longitudinal apertures are in themselves of lesser interest than the results, and since the derivations do not contain any new features and are somewhat lengthy, they are not included here.

Several applications of the small longitudinal aperture and longitudinal obstacle expressions presented in Section II are also included here. These applications serve partly to illustrate the use of these relations, but are also of interest in themselves and were solved originally in answer to a need. In Section III, the small aperture formulas are applied first to a round hole located in the center fin of trough waveguide, and then to an array of holes coupling two parallel plate waveguides. It is pointed out there that the latter case is of particular value in its use in a transverse resonance procedure, and can be applied, for example, to top wall directional couplers in rectangular waveguide, or to a periodic array of holes in trough waveguide. The small obstacle formulas are applied in Section IV first to a circular disk in rectangular waveguide, where the distinctions between longitudinal and transverse orientations of the disk are determined, and then to the case of a plane wave incident at an angle on a two-dimensional array of longitudinal rods. The latter problem is of interest in a study of anisotropic radomes, since this configuration discriminates between incident waves of parallel and perpendicular polarization.

II. Small Aperture and Small Obstacle Formulas

A. Preliminary Relations and Definitions of Terms

The incident mode in the waveguide containing the aperture or obstacle may be characterized by transverse mode functions $\mathbf{e}$ and $\mathbf{h}$ and by a longitudinal scalar function ϕ or ψ. These functions and their normalizations are the same as those in the "Waveguide Handbook;"[8] expressions for these functions for conventional waveguides are also presented there.[9] The scalar function ϕ and ψ are proportional to longitudinal components of electric and magnetic field, respectively, so that $\phi=0$ for H (or TE) modes and $\psi=0$ for E (or TM) modes. When the mode functions and the scalar functions are normalized in the manner

$$\iint_{\text{cross section}} \mathbf{e}\cdot\mathbf{e}^* dS = \iint_{\text{cross section}} \mathbf{h}\cdot\mathbf{h}^* dS = 1, \qquad (1)$$

and

$$\begin{aligned} \iint_{\text{cross section}} \phi\phi^* dS &= 1/k_c^2, \\ \iint_{\text{cross section}} \psi\psi^* dS &= 1/k_c^2, \end{aligned} \qquad (2)$$

[1] H. A. Bethe, "Lumped Constants for Small Irises," M.I.T. Rad. Lab., Cambridge, Mass., Rept. No. 43-22; March, 1943.

[2] H. A. Bethe, "Theory of Side Windows in Waveguides," M.I.T. Rad. Lab., Cambridge, Mass., Rept. No. 43-27; April, 1943.

[3] N. Marcuvitz, "Waveguide Circuit Theory: Coupling of Waveguides by Small Apertures," Microwave Res. Inst., Polytechnic Inst. of Brooklyn, Rept. No. R-157-47, PIB-106; 1947.

[4] H. A. Judy and D. J. Angelakos, "Mode Selective Directional Couplers," Electronics Res. Lab., University of California, Berkeley, Ser. No. 60, Issue No. 19; September, 1954.

[5] L. B. Felsen, "Analysis of Circular Waveguide Modes," Microwave Res. Inst., Polytechnic Inst. of Brooklyn, Second Quarterly Rept., R-394.6-55, PIB-327.6; February, 1955.

[6] N. Marcuvitz, "Waveguide Handbook," Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 10; 1951. See for example, Sections 6.6, 6.8–6.10, 7.2–7.5.

[7] L. B. Felsen of the Polytechnic Institute of Brooklyn has independently derived a small obstacle formulation which applies also to unsymmetrical obstacles.

[8] Marcuvitz, *op. cit.*, Section 1.2.

[9] *Ibid*, Chap. II.

where $k_c(=2\pi/\lambda_c)$ is the cutoff wave number of the given mode; then the characteristic impedance Z_0 and the characteristic admittance Y_0 are equal to the wave impedance and admittance, respectively, *i.e.*,

$$Z_0 = 1/Y_0 = \begin{cases} \dfrac{\kappa}{\omega\epsilon} & \text{for E modes} \\ \dfrac{\omega\mu}{\kappa} & \text{for H modes} \end{cases} \tag{3}$$

where $\kappa(=2\pi/\lambda_g)$ is the propagation wave number.

It is convenient to employ the symbols e_z and h_z to represent the longitudinal field components; their relation to ϕ and ψ are

$$e_z = -j\frac{Y_0}{\omega\epsilon}k_c^2\phi \tag{4}$$

$$h_z = -j\frac{Z_0}{\omega\mu}k_c^2\psi. \tag{5}$$

Since the incident mode field does not vary much over a small aperture or obstacle, the value of the incident field is taken to be that at the center of the aperture or obstacle, and this particular value of the field is denoted by the subscript 0.

In small aperture theory, the behavior of the aperture is dependent on the magnetic and electric polarizabilities, M and P, which respond, respectively, to the tangential component(s) of magnetic field and the normal component of electric field at the aperture. These polarizabilities are a function of the shape and size of the aperture only, and expressions are available for the polarizabilities of apertures of simple shapes, such as a circle, ellipse, or long slit.[10] Numerical values for other shapes have been obtained experimentally by Cohn[11,12] using an electrolytic tank, thereby enhancing the usefulness of small aperture theory. Cohn has also derived a simple correction formula[13] for the magnetic polarizability in the case of larger apertures.

The behavior of a small obstacle is sensitive to the tangential component(s) of electric field and the normal component(s) of magnetic field at the obstacle. The respective polarizabilities are designated in this paper as P^{ob} and M^{ob}. For a planar obstacle, the numerical values of P^{ob} and M^{ob} are equal to those of M and P, respectively, for an aperture identical in size and shape to the obstacle. Such an equivalence is generally deduced via Babinet equivalence considerations, involving a factor of 4, but we choose to place this factor in the expressions for the equivalent circuit parameters rather than in the polarizability expressions directly. Expressions for certain nonplanar obstacles of simple shape are scattered throughout the literature.[14,15]

Since the apertures and obstacles treated in this paper are all symmetrical and lossless, only two independent parameters are required for their circuit representation. The reactance and susceptance representations are shown pictorially by the tee and pi networks of Fig. 1. The parameters are related to the reactance and susceptance matrix elements by

$$X_a = X_{11} - X_{12},\ X_b = X_{12}$$
$$B_a = B_{11} - B_{12},\ B_b = B_{12}. \tag{6}$$

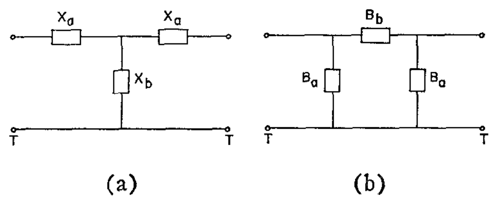


Fig. 1—(a) Tee equivalent network. (b) Pi equivalent network.

In the small aperture or obstacle limit, the reactance and susceptance parameters are very simply related, *i.e.*,

$$X_a' = -\frac{1}{2B_b'}$$
$$X_b' = -\frac{1}{2B_a'}, \tag{7}$$

where the prime denotes the parameter normalized to the appropriate characteristic impedance or admittance. Relations (7) state in essence that the shunt effect and the series effect of the networks are independent in this limit.

In a scattering matrix representation of a symmetrical aperture or obstacle, only parameters S_{11} and S_{12} are independent. Furthermore, one can write for small apertures or obstacles

$$S_{12} = 1 + \sigma. \tag{8}$$

Element S_{12} should actually be expressed as

$$S_{12} = \frac{1}{1-\sigma},$$

so that $|S_{12}|^2 < 1$, but in the small aperture or obstacle limit $\sigma \ll 1$. Elements S_{11} and σ are related to the pi network parameters in this limit as

$$jB_b' = \frac{1}{S_{11} - \sigma}$$
$$jB_a' = -\tfrac{1}{2}(S_{11} + \sigma), \tag{9}$$

[10] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, "Principles of Microwave Circuits," Rad. Lab. Sers., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 8, p. 178; 1948.

[11] S. B. Cohn, "Determination of aperture parameters by electrolytic-tank measurements," Proc. IRE, vol. 39, pp. 1416–1421; November, 1951.

[12] S. B. Cohn, "The electric polarizability of apertures of arbitrary shape," Proc. IRE, vol. 40, pp. 1069–1071; September, 1952.

[13] S. B. Cohn, "Microwave coupling by large apertures," Proc. IRE, vol. 40, pp. 696–699; June, 1952.

[14] C. Susskind, "Obstacle type artificial dielectrics for microwaves," *J. Brit. IRE*, vol. 12, p. 49; 1952.

[15] M. M. Z. Kharadly and W. Jackson, "The properties of artificial dielectrics comprising arrays of conducting elements," *Proc. I.E.E.* (London), Pt. III, vol. 100, pp. 199–212; July, 1953.

or

$$S_{11} = -j\left[B_a' + \frac{1}{2B_b'}\right]$$

$$\sigma = -j\left[B_a' - \frac{1}{2B_b'}\right]. \qquad (10)$$

B. Symmetrical Longitudinal Apertures Coupling Identical Waveguides

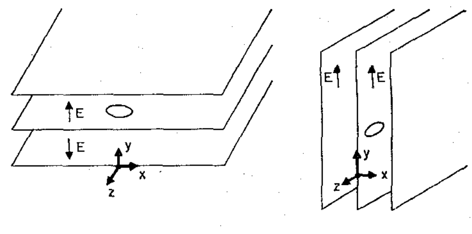
Longitudinal apertures may be used to couple two waveguides either in a tee junction fashion or when placed parallel to each other. In either case, an appropriate equivalent circuit representation is chosen, and by small aperture theory the parameters of the equivalent circuit are related to the geometry of the coupling aperture and the coupled waveguides. Equivalent circuits for several specific examples of these two types are presented by Marcuvitz.[6]

When the two coupled waveguides are identical and are arranged parallel to each other, the form of the equivalent circuit becomes particularly simple. This special case corresponds to many practical situations; it occurs, for example, in various directional coupler applications and in strip line and trough waveguide discontinuities. The longitudinal aperture discussion in this paper is restricted to this class of structures.

Two typical cases which arise in the coupling of two parallel identical waveguiding regions are illustrated in Fig. 2. The coupling behavior for arbitrary field excitation may be expressed in terms of two orthogonal modal situations, one for which the excitations from the two separate waveguides are in phase and the second for which they are out of phase. When the field excitations in the separate waveguides are opposite to those indicated in Fig. 2, the surface common to the two identical waveguide regions becomes an electric wall and the presence of the coupling aperture is not felt. The equivalent circuit for this excitation degenerates into a straight-through connection for each waveguide separately, with no coupling between them.

When the excitations in the separate waveguides are as shown in Fig. 2, the aperture surface becomes a magnetic wall and the equivalent circuit is now nontrivial. Since the two halves of these structures are identical to each other, the reflection coefficients for each half are equal and are the same as the reflection coefficient for both halves taken together. The complete electrical behavior for this excitation is thus obtained by choosing a pi or tee network representation, of the form of Fig. 1, for either half of the structure.

It may be desirable to have available a single equivalent circuit appropriate to both excitations, or, equivalently, to an arbitrary excitation. In that case, a form such as that shown in Fig. 3 may be used; as seen, the circuit consists of two pi networks, back to back, bridged across by an additional element equal to $-B_a$. When the excitation is such as to produce a magnetic wall in the



(a) (b)

Fig. 2—Longitudinal aperture coupling of identical waveguides. (a) E plane (top wall) coupling. (b) H plane (side wall) coupling.

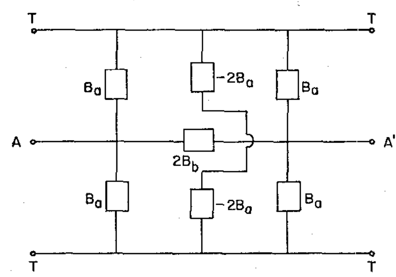


Fig. 3—Equivalent circuit for longitudinal aperture coupling two identical waveguides, valid for arbitrary excitation.

aperture, an open circuit is produced along the line $A-A'$ in Fig. 3, so that the circuit breaks up into two separate pi networks, back to back, and element $-B_a$ does not contribute. When the excitation is such as to produce an electric wall in the aperture, a short circuit occurs along $A-A'$ and the additional element $-B_a$ cancels the remaining elements to produce straight-through connections in the separate waveguides.

E Plane (Top Wall) Coupling: The structure appropriate to this situation was indicated in Fig. 2(a). The waves in both of the constituent waveguides propagate along the z direction, and the principal axes of the symmetrical coupling aperture are assumed to lie along the x and z axes. In small aperture theory, the coupling by the aperture is sensitive to the H_x, H_z, and E_y components of the unperturbed fields in the constituent waveguides. The expressions for the equivalent circuit parameters will therefore contain the M_x, M_z, and P_y polarizabilities in the general case. Only two parameters, B_a and B_b, are needed to specify the complete equivalent network, either in pi form for the "magnetic wall" excitation or in the form of Fig. 3 which is valid for arbitrary excitation. It can be shown that small aperture expressions for these two parameters, normalized to the characteristic admittance Y_0 of either of the identical coupled waveguides, are

$$\frac{1}{B_b'} = -\omega\mu Y_0 2M_x h_{x0} h_{x0}^* \qquad (11)$$

$$B_a' = \omega\mu Y_0 M_z h_{z0} h_{z0}^* - \omega\epsilon Z_0 P_y e_{y0} e_{y0}^*. \qquad (12)$$

The polarizabilities M and P have been discussed in the previous section, Y_0 and Z_0 are defined in (3), and the mode functions are defined with respect to their normalizations in (1), (2), (4), and (5). The subscript 0 denotes the value of the mode function at the center of the aperture, and the asterisk means the complex conjugate. In the normalizations defined by (1) and (2), the integrations are performed over *either* of the two identical waveguides, but not over both. One sees from (11) and (12) that B_b' is always inductive, while B_a' can be either inductive or capacitive.

H Plane (Side Wall) Coupling: A structure corresponding to coupling of this type is shown in Fig. 2(b). Again, the power flow in both of the constituent waveguides is in the z direction, and the principal axes of the coupling aperture lie along the y and z directions. In the most general case, which might arise for some higher mode, the coupling would be responsive to the H_y, H_z, and E_x components of the unperturbed fields in the separate waveguides. For this case the expressions for B_a' and B_b' would be given by (11) and (12) upon replacement of M_x and P_y by M_y and P_x. Often, however, this type of coupling occurs when E_x and H_y are both zero in the unperturbed waveguides. Then (11) and (12) reduce to

$$\frac{1}{B_b'} \approx 0 \tag{13}$$

$$B_a' = \omega\mu Y_0 M_z h_{z0} h_{z0}^*. \tag{14}$$

It is seen from (13) and (14) that *to this order* the equivalent circuit reduces to a simple shunt capacitance. The other elements are actually nonvanishing but generally can be neglected; they correspond to higher order (multipole) contributions.[16]

C. Symmetrical Longitudinal Obstacles

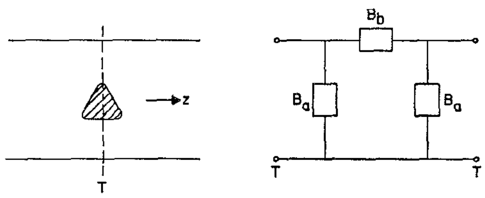
A typical small symmetrical longitudinal obstacle and an equivalent circuit for it in pi form are shown in Fig. 4(a) and (b). Since the obstacle is symmetrical, only two parameters suffice to characterize it completely. In small obstacle theory, the electrical behavior of the obstacle is responsive to the tangential component(s) of electric field and the normal component(s) of magnetic field at the obstacle. As mentioned in Section II-A, the respective obstacle polarizabilities are designated in this paper as M^{ob} and P^{ob}.

The direction of propagation in the waveguide is denoted by z, with x and y referring to the cross-section coordinates. With these coordinates, small obstacle expressions for the parameters of the equivalent network of Fig. 4(b) may be shown to be

$$B_a' = 2\omega\epsilon Z_0[M_x^{ob} e_{x0} \overset{*}{e}_{x0} + M_y^{ob} e_{y0} \overset{*}{e}_{y0}] - 2\omega\mu Y_0[P_z^{ob} h_{z0} \overset{*}{h}_{z0}] \tag{15}$$

$$\frac{1}{B_b'} = 4\omega\mu Y_0[P_x^{ob} h_{x0} \overset{*}{h}_{x0} + P_y^{ob} h_{y0} \overset{*}{h}_{y0}] - 4\omega\epsilon Z_0[M_z^{ob} e_{z0} \overset{*}{e}_{z0}]. \tag{16}$$

[16] For example, see N. Marcuvitz, *op. cit.*, pp. 379 (2), and 380 (6).



(a) (b)

Fig. 4—Symmetrical longitudinal obstacle. (a) Geometry. (b) Typical equivalent circuit.

The various quantities appearing in (15) and (16) have been defined and discussed in Section II-A. In particular, attention is called to the remarks concerning the polarizabilities. The transverse mode functions $\boldsymbol{h}$ and $\boldsymbol{e}$ have been discussed above in their vector form; the components employed in (15) and (16) are related to the vector form in the evident manner

$$\boldsymbol{h} = h_x \mathbf{x}_0 + h_y \boldsymbol{y}_0, \; \boldsymbol{e} = e_x \mathbf{x}_0 + e_y \boldsymbol{y}_0, \tag{17}$$

where $\mathbf{x}_0$ and $\boldsymbol{y}_0$ are unit vectors.

Since expressions (15) and (16) are valid for any small symmetrical, but otherwise general, obstacle, they should also apply to a transverse planar obstacle of the type shown in Fig. 5(a). The equivalent circuit for this transverse obstacle is purely shunt, as seen in Fig. 5(b). Since the obstacle is now responsive to E_x, E_y, and H_z, only the M_x^{ob}, M_y^{ob}, and P_z^{ob} polarizability components will be nonvanishing. One then finds from (15) and (16) for a *transverse planar obstacle:*

$$\frac{1}{B_b'} = 0,$$

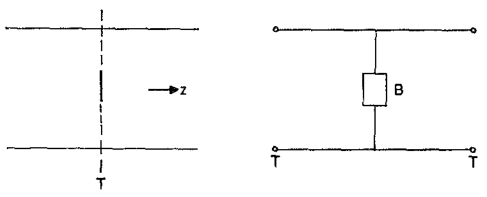
$$B' = 2B_a' = 4\omega\epsilon Z_0[M_x^{ob} e_{x0} e_{x0}^* + M_y^{ob} e_{y0} e_{y0}^*] - 4\omega\mu Y_0[P_z^{ob} h_{z0} h_{z0}^*]. \tag{18}$$

Result (18) is simply a rephrasing in the notation of this paper of the well-known result for a transverse planar obstacle. The factor of 4 arises because the numerical values for M^{ob} and P^{ob} are equal to those of M and P, respectively, for a transverse aperture identical in size and shape to the obstacle.

III. Applications of Small Aperture Formulas

A. Circular Hole in Trough Waveguide

The small aperture expressions are here employed to evaluate the equivalent circuit parameters of a circular hole located in the center fin of trough waveguide. The geometry is illustrated in Fig. 6(a). For the round hole this type of calculation is usually very good except when the hole is very large or almost in contact with the side wall, or is near to the edge of the fin. Since the trough waveguide is symmetrical, and therefore nonradiating, and since the hole is located on the center fin, the equivalent circuit for the hole is purely reactive. Due to the shape of the hole, the circuit is also symmetrical, and may be chosen in the form of the pi network of Fig. 6(b). It will be seen, as implied by Fig. 6(b), that the series element B_b is always inductive



(a) (b)

Fig. 5—Planar transverse obstacle. (a) Geometry. (b) Equivalent circuit.

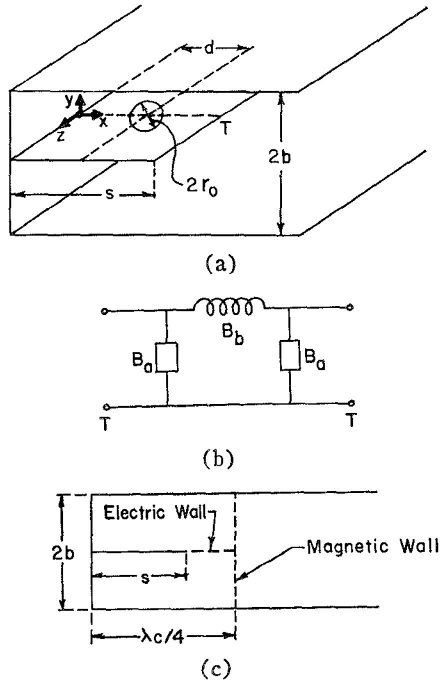


Fig. 6—Circular hole in center fin of trough waveguide. (a) Geometry. (b) Equivalent circuit. (c) Rectangular waveguide approximation to trough waveguide.

while the shunt arms B_a may be inductive or capacitive depending on the location of the hole (value of d).

The relation between the electric fields in the upper and lower portions of the trough waveguide in its usual mode of operation is that indicated in Fig. 2(a). Consequently, (11) and (12) are the appropriate expressions to use for the determination of B_a' and B_b'. Before these expressions can be applied, however, one must have knowledge of the polarizabilities M and P, the quantities Y_0 and Z_0, and expressions for the mode functions.

For the case of a circular hole, the polarizabilities are

$$M_x = M_z = \frac{4}{3} r_0^3$$

$$P_y = \frac{2}{3} r_0^3, \tag{19}$$

where r_0 is the radius of the hole. Since the dominant mode in trough waveguide is an H (or TE) mode, the appropriate relations for Y_0 and Z_0 are, from (3),

$$Z_0 = \frac{1}{Y_0} = \frac{\omega\mu}{\kappa}, \tag{20}$$

where $\kappa(=2\pi/\lambda_g)$ is the propagation wave number.

Because the rigorous mode functions for trough waveguide are somewhat involved, a simple approximation is employed here which is expected to be quite accurate. The lumped effect of the field distribution away from the edge of the fin can be very well approximated by replacing the fin edge with its associated fringing field by an additional width of fin and a magnetic wall (open circuit) at the end of this extension, as shown in Fig. 6(c). The structure then becomes two half-sections of rectangular waveguide, one on top of the other, coupled by the circular hole. The amount by which the center fin is extended can be obtained from the knowledge of the available value for the cutoff wavelength,[17,18] since, as shown in Fig. 6(c), the original fin width plus the extension must be equal to $\lambda_c/4$.

The mode functions h and e can now very readily be obtained from the equivalence of Fig. 6(c) and the normalization relations (1) and (2). Noting that the origin of coordinates is taken at the junction of the fin with the side wall, we find from an integration over one of the two halves of the structure that relations (1) and (2) yield

$$e(x) = h(x) = 2\sqrt{\frac{2}{b\lambda_c}} \sin\left(\frac{2\pi x}{\lambda_c}\right) \tag{21}$$

$$\psi(x) = \frac{\lambda_c}{\pi}\sqrt{\frac{2}{b\lambda_c}} \cos\left(\frac{2\pi x}{\lambda_c}\right), \tag{22}$$

using $\lambda_c = 2\pi/k_c$.

The equivalent circuit parameters for the circular hole can now be found by employing relations (19)–(22) and (5) in expressions (11) and (12). Noting that the center of the hole is located at $x=d$, we obtain after simplification

$$B_b' = -\frac{3b\lambda_g}{k_c(4r_0)^3 \sin^2 k_c d} \tag{23}$$

$$B_a' = \frac{\lambda_g(2k_c r_0)^3 \cos^2 k_c d}{3\pi^2 b}\left[1 - \frac{1}{2}\left(\frac{k}{k_c}\right)^2 \tan^2 k_c d\right] \tag{24}$$

where $k_c = 2\pi/\lambda_c$, $k = 2\pi/\lambda$. We also note that k must exceed k_c for propagation. Results (23) and (24) were originally derived as byproducts in an analysis of periodic structures in trough waveguide.[19]

From (24), one sees that B_a' can be capacitive or inductive, depending upon the frequency and the location of the hole. The inductive contribution is greater if the hole is located nearer to the fin edge. For an appropriate hole location, $B_a' = 0$, and the equivalent circuit be-

[17] A. A. Oliner, "Theoretical developments in symmetrical strip transmission line," Proc. Symp. on Modern Advances in Microwave Techniques, Polytechnic Institute of Brooklyn, Brooklyn, N. Y., pp. 387–390; November, 1954.

[18] K. S. Packard, "The cutoff wavelength of trough waveguide," IRE Trans., vol. MTT-6, pp. 455, 456; October, 1958.

[19] A. A. Oliner and W. Rotman, "Periodic structures in trough waveguide," IRE Trans. on Microwave Theory and Techniques, vol. MTT-7, pp. 134–140, (3) and (4); January, 1959.

comes a *pure series inductance.* This condition is, of course, given by

$$k = \sqrt{2}k_c \cot k_c d, \tag{25}$$

or

$$d = \frac{1}{k_c} \cot^{-1}\left(\frac{k}{\sqrt{2}k_c}\right). \tag{26}$$

We also note that it is possible at a given frequency or for a given hole location to obtain a *reflectionless* discontinuity. From (10), *i.e.*, from $S_{11}=0$, we see that a unity VSWR occurs when

$$k = \sqrt{\frac{2}{3}}\,\frac{k_c}{\sin k_c d} \tag{27}$$

or, alternatively, when

$$d = \frac{1}{k_c}\sin^{-1}\left[\sqrt{\frac{2}{3}}\,\frac{k_c}{k}\right]. \tag{28}$$

At these values of k or d the hole introduces only phase shift.

Numerical values for a typical case are presented in Fig. 7. The following dimensions [see Fig. 6(a)] have been taken: $2b=1.00$ inch, $s=1.00$ inch, $r_0=0.25$ inch, $\lambda=3.50$ inches; the wavelength chosen corresponds roughly to midband operation. Fig. 7 presents B_a' and B_b' as a function of the location of the hole on the fin. For these dimensions, one finds $d=0.617$ inch and $d=0.485$ inch to be the hole locations such that the equivalent circuit is pure series and the hole is nonreflecting, respectively. Fig. 7 also includes a curve of VSWR vs hole location.

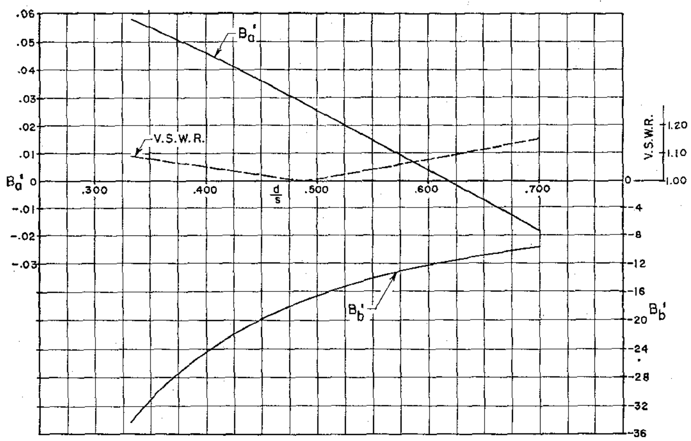


Fig. 7—Network parameter values for circular hole in trough waveguide.

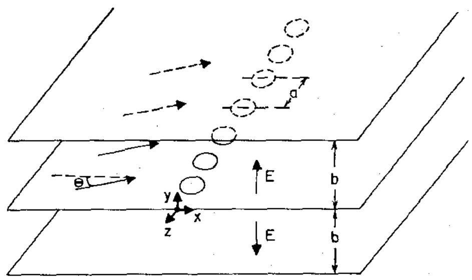


Fig. 8—Wave incident at angle θ on array of holes coupling parallel plate waveguides.

B. Array of Holes Coupling Parallel Plate Waveguides

In this section, small aperture expressions are employed to obtain the equivalent circuit parameters for an array of holes which couples two identical parallel plate waveguides. The geometry of the configuration is shown in Fig. 8. As shown, the waves in the parallel plate guides have oppositely directed electric fields and are incident on the array of holes at an angle θ with respect to the x direction. Because of the symmetry of the structure and the excitation, the equivalent circuit can be expressed in pi form and the parameters can be determined by the use of (11) and (12). The equivalent circuit, in fact, will be seen to be of the form of Fig. 6(b), with the series element always inductive and the shunt elements capacitive or inductive, depending on the angle θ of incidence.

The real value in the solution of this problem lies not in the direct phrasing of it, as given above, but in its application to the transverse resonance analysis of a number of waveguiding structures possessing a longitudinal array of holes. For example, if metal plates are placed at the sides of the structure in Fig. 8, one has a *top wall directional coupler in rectangular waveguide.* The use in a transverse resonance procedure of the equivalent circuit for the array of holes permits the determination of the properties of the coupler. A second example is an *array of longitudinal holes in trough waveguide.* Again, the results of this section are employed in a transverse resonance procedure.[20]

The TEM waves incident at angle θ are viewed in this analysis as being equivalent to H (or TE) waves incident normally (in the x direction) on the array of holes. The use of this technique simplifies the determination of the mode functions. The propagation wave number κ is related to angle θ as

$$\kappa = k\cos\theta, \tag{29}$$

and all field components experience an exponential variation of the form $\exp(-jk_z z)$, where

$$k_z = k\sin\theta. \tag{30}$$

Due to the periodicity of the array of holes, the modes in the vicinity of the array possess fields which are also periodic.[21] When the holes are sufficiently closely spaced together, the higher modes are nonpropagating. The mode functions for the dominant mode, normalized to a unit cell of the array, are then obtained by integrating over a unit cell in either the upper or lower portion of the configuration in the manner of (1) and (2). One finds, as a result,

[20] *Ibid.*, see Section II-B., pp. 137–138.
[21] Marcuvitz, *op. cit.*, pp. 88, 89.

$$h_z(z) = e_y(z) = \frac{1}{\sqrt{ab}} e^{-jk_z z}, \quad (31)$$

$$\psi(z) = \frac{1}{k_z\sqrt{ab}} e^{-jk_z z}, \quad (32)$$

where $k_c = k_z$. On use of (5) and (20), since the dominant mode is an H (or TE) mode, (31) and (32) become

$$h_z h_z^* = e_y e_y^* = 1/ab \quad (33)$$

$$h_x h_x^* = \left(\frac{k_z}{\kappa}\right)^2 1/ab. \quad (34)$$

When (33), (34), and (20) are used in relations (11) and (12), with x and z interchanged, expressions for the pi network parameters become

$$B_b' = -\frac{ab}{2\kappa M_z} \quad (35)$$

$$B_a' = \frac{1}{ab\kappa}[M_x k_z^2 - P_y k^2], \quad (36)$$

where M_x, M_z, and P_y are the magnetic and electric polarizabilities for an individual hole. In some cases, close proximity of the holes to each other may produce mutual coupling effects which will alter the polarizability values.

When the problem is phrased in terms of waves incident at angle θ, (35) and (36) become, on use of (29) and (30),

$$B_b' = -\frac{ab}{2kM_z\cos\theta} \quad (37)$$

$$B_a' = \frac{k}{ab\cos\theta}[M_x\sin^2\theta - P_y]. \quad (38)$$

One sees from (38) that B_a' can be either capacitive or inductive depending on θ.

When the equivalent circuit is to be used in a transverse resonance context, propagation actually occurs in the z direction so that in (35) and (36) k_z become the propagation wave number ($=2\pi/\lambda_g$) and κ becomes the transverse wave number k_t. Moreover, to be useful in a transverse resonance context, (35) and (36) must involve k_t but not k_z. Since k_t and k_z are related via

$$k^2 = k_t^2 + k_z^2,$$

(35) and (36) become

$$B_b' = -\frac{ab}{2k_t M_z} \quad (39)$$

$$B_a' = \frac{1}{abk_t}[(M_z - P_y)k^2 - M_z k_t^2]. \quad (40)$$

It is significant that B_a' involves the free space wave number k in addition to k_t. As a result, the transverse wave number will be frequency dependent, a property characteristic of periodic structures.

IV. Applications of Small Obstacle Formulas

A. Circular Disk in Rectangular Waveguide

As a simple illustration of the use of small obstacle expressions, the equivalent circuit of a centered circular metallic disk in rectangular waveguide can be examined for different orientations of the disk. For example, consider the disk located longitudinally and then transversely, as shown in Figs. 9(a) and (b).

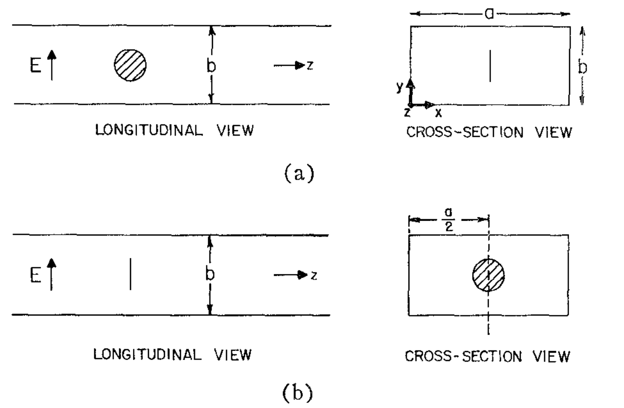


Fig. 9—Circular disc in rectangular waveguide. (a) Longitudinally oriented disc. (b) Transversely oriented disk.

The polarizabilities of the circular disk are

$$M_y^{ob} = M_z^{ob} = \frac{4}{3} r_0^3 \quad (41)$$

$$P_x^{ob} = \frac{2}{3} r_0^3 \quad (42)$$

for the longitudinal orientation; for the transverse orientation of Fig. 9(b), M_z^{ob} and P_x^{ob} should be replaced by M_x^{ob} and P_z^{ob}, respectively. The dominant mode in rectangular waveguide is an H (or TE) mode, so that its characteristic impedance has the form (20); the corresponding transverse mode functions are[22]

$$h_x(x) = e_y(x) = \sqrt{\frac{2}{ab}} \sin\frac{\pi x}{a}. \quad (43)$$

The longitudinal magnetic field has a cos $(\pi x/a)$ dependence, and is therefore zero at the center of the disk and can be neglected.

The parameters B_a' and B_b' of the pi network equivalent [see Fig. 4(b)] of the longitudinally oriented circular disk are found by employing (41), (42), (43), and (20) in relations (15) and (16). One finds then that

$$B_a' = \frac{4\pi}{3}\frac{(2r_0)^3}{ab}\frac{\lambda_g}{\lambda^2} \quad (44)$$

$$\frac{1}{B_b'} = \frac{4\pi}{3}\frac{(2r_0)^3}{ab\lambda_g}. \quad (45)$$

[22] *Ibid.*, Section 2.2.

When these same expressions are inserted into (18) one obtains the following result for the shunt element B' [see Fig. 4(a)] of the transversely oriented circular disk:

$$B' = \frac{8\pi}{3} \frac{(2r_0)^3}{ab} \frac{\lambda_g}{\lambda^2} . \tag{46}$$

When the two equivalent circuits are compared, it is seen that the total shunt effect of each is identical, due to the electric field component parallel to the disk, while for the longitudinal orientation an additional series capacitive element is present, resulting from the component of magnetic field normal to the disk. Since the shunt and series elements have the same sign, one sees from relation (10) that the VSWR introduced by the longitudinal orientation is greater than that produced by the transverse orientation.

B. *Array of Longitudinal Rods in Free Space*

The infinite two-dimensional array of longitudinal rods to be analyzed is described in Fig. 10. A plane wave of so-called parallel polarization is shown in Fig. 10(a) to be incident on the array of rods at angle θ with respect to the z direction. With such a polarization a component of electric field is set up parallel to the conducting rods, so that the rods exert a significant effect on the wave. For perpendicular polarization, on the other hand, a component of magnetic field would be created parallel to the rods and the wave would be negligibly affected if the rods were thin. Such a longitudinal array of rods thus serves to discriminate between the two polarizations.

For the polarization shown in Fig. 10(a), the total field consists of H_y, E_x, and E_z components. The plane wave incident at angle θ thus may be viewed as an E (or TM) mode incident along the z direction, with characteristic impedance Z_0 given by (3) as

$$Z_0 = \frac{1}{Y_0} = \frac{\kappa}{\omega\epsilon}, \tag{47}$$

with

$$\kappa = k \cos \theta. \tag{48}$$

When the spacing between successive rods is less than half a free-space wavelength, all the higher modes are nonpropagating. The mode functions of the dominant mode, which is the incident wave, normalized to the unit cell of dimensions a by b, are found by integrating over the unit cell according to (1) and (2). On use of (4), the mode functions may be written as

$$e_x(x) = h_y(x) = \frac{1}{\sqrt{ab}} e^{-jkx \sin \theta} \tag{49}$$

$$e_z(x) = -j \frac{\tan \theta}{\sqrt{ab}} e^{-jkx \sin \theta} \tag{50}$$

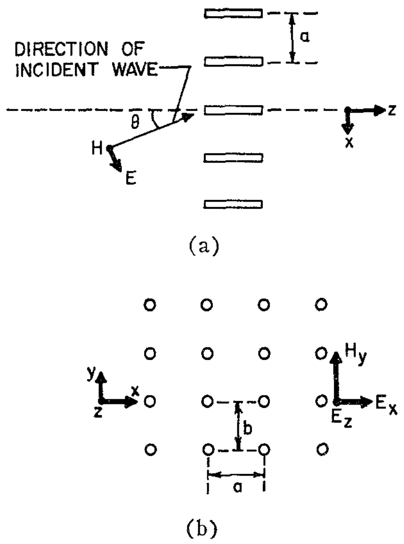


Fig. 10—Plane-wave incident on infinite two-dimensional array of longitudinal rods in free space. (a) Side view. (b) Cross-section view.

since

$$k_c = k_x = k \sin \theta. \tag{51}$$

For simplicity, we assume that the rods are thin so that the only non-negligible polarizability is M_z^{ob}, *i.e.*, the rods are sensitive only to the longitudinal component of electric field. Under these conditions, the equivalent circuit parameters can be evaluated via expressions (15) and (16), using (47), (49), and (50). One finds that[23]

$$B_a' = 0$$

$$-\frac{1}{B_b'} = X' = \frac{8\pi M_z^{ob}}{ab\lambda} \frac{\sin^2 \theta}{\cos \theta} . \tag{52}$$

Result (52) states that the equivalent circuit consists only of a series inductance, the value of which is proportional to the polarizability. It is of interest that the element is inductive rather than capacitive, since an array of similar rods transversely oriented would be characterized by a shunt capacitive equivalent circuit. An analogous situation arises in connection with slots whose length is smaller than that required for resonance. Such a transverse slot in rectangular waveguide is inductive, while the equivalent circuit for this slot cut in the top wall of rectangular waveguide (a "longitudinal shunt slot") is a shunt capacitance.

ACKNOWLEDGMENT

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[23] This result was obtained by the writer for the Technical Research Group in 1955 as part of consulting services on anisotropic radomes.